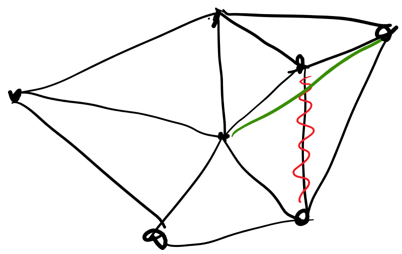
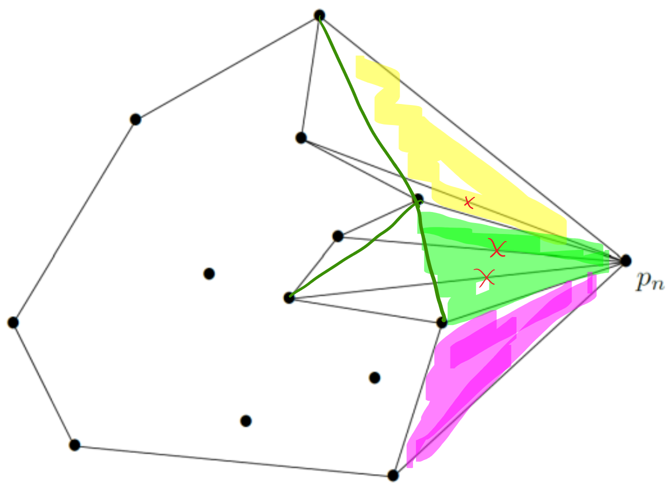
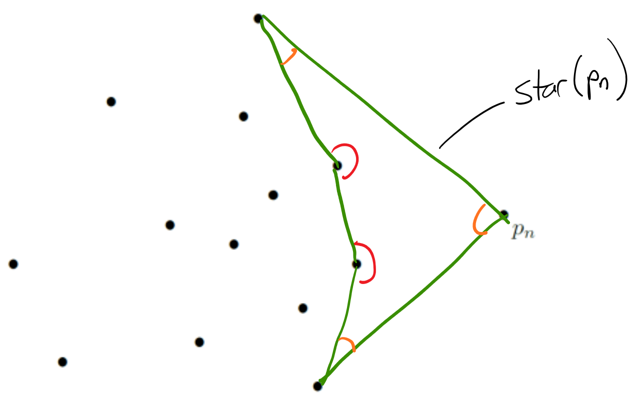
Triangulations of point sets



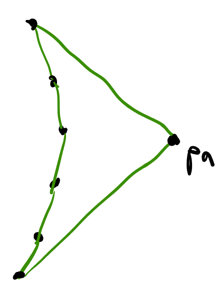
An edge flip converts one triangulation into another.

If I have triangulations T_1 and T_2 of the same point set, can I find a sequence of edge flips to transform T_1 into T_2?



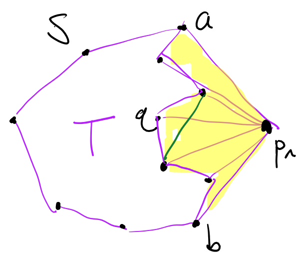


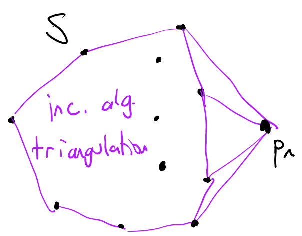
STAR of a vertex: the union of all triangles incident to that vertex



3 convex angles in star(p_n)

LEMMA: Let $S = \{p_1, \dots, p_n\}$ in order of incr. x-coord, and let T be any triangulation of S. We can transform star(p_n) in T to the set of triangles that contain p_n and result from the incremental alg. by a sequence of edge flips.





proof: Let a and b be the convex hull vertices adjacent to p_n.

We want a and b to be connected by a chain of reflex vertices on the left.

(*) If I find a convex vertex q in the chain from a to b, then q is a diagonal of a convex quadrilateral. So flip edge $\overline{qp_n}$.

This decreases degree of p_n by 1.

Repeat this process (*), which must terminate since the degree of p_n must be at least 2.

When there are no more convex vertices in the chain from a to b, then we have achieved the set of triangles desired.

THEOREM: The flip graph of any point set in the plane is connected.

proof: It suffices to show that any triangulation can be transformed into the triangulation that results from the inc. alg. by edge flips.

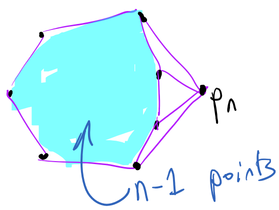
Use induction on the number of points n.

base case: If $n = 3$, then there is only one triangulation.

induction: Assume the flip graph is connected whenever S has fewer than n points.

Let $S = \{p_1, p_2, \ldots, p_n\}$ and T is any triangulation of S.

By the lemma, the $\mathrm{star}(p_n)$ in T can be converted to the set of triangles incident to p_n arising from the inc. alg.



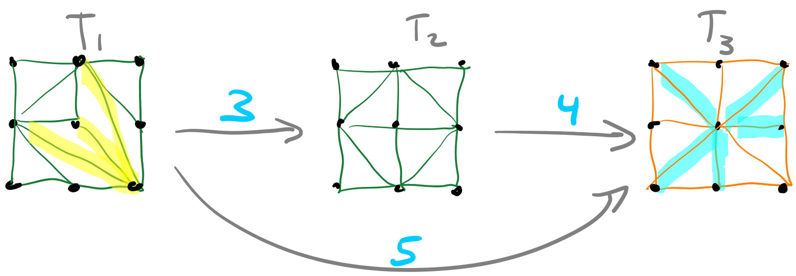
Consider $S \setminus \{p_n\}$, which is a set of $n-1$ points.

set difference: S without the point p_n

By the inductive hypothesis, $S \setminus \{p_n\}$ can be transformed via edge flips into its inc. alg. triangulation.

Then reattach $\mathrm{star}(p_n)$ from the lemma, and we obtain the inc. alg. triangulation of S.

How many edge flips may be required to transform one triangulation into another?



COROLLARY: For a planar point set S of n points, the diameter of its flip graph is at most $(n-2)(n-3)$.

length of the shortest path between the most distant nodes.

proof: In the lemma above, at most $n-3$ edge flips are required.

In the inductive argument, the number of vertices decreases by 1, so does the max number of edge flips.

The total max number of edge flips:

$$\underset{1}{(n-3)} + \underset{2}{(n-4)} + \underset{3}{(n-5)} + \cdots + \underset{(n-5)}{3} + \underset{(n-4)}{2} + \underset{(n-3)}{1} = \frac{(n-3)(n-2)}{2}$$

The diameter of the flip graph is at most double this, so $(n-3)(n-2)$.

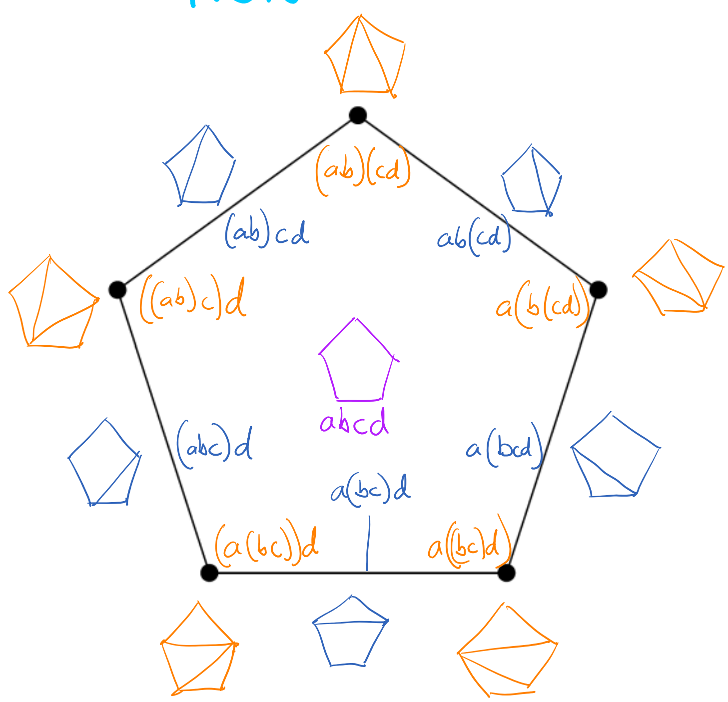
example: If $n=9$, then diameter of flip graph is $\leq 6 \cdot 7 = 42$

3D or higher?

In dimension 5 or higher, the flip graph may be disconnected.

In dimensions 3 and 4, it is not known whether the flip graph is connected.

THE ASSOCIAHEDRON



ASSOCIATIVE:

((a b) c) d   (a (b c)) d   (a b)(c d)

a (b (c d))   a ((b c) d)

(a b)(c d) e) — 14 ways to do it