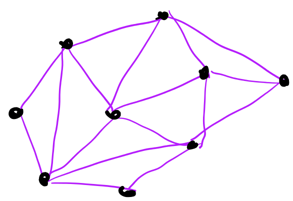
TRIANGULATIONS OF POINTS IN A PLANE

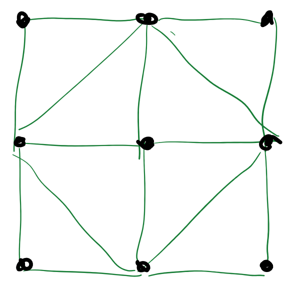
A **triangulation** of a planar point set S is a subdivision of the plane determined by a maximal set of noncrossing edges whose vertex set is S.

example:

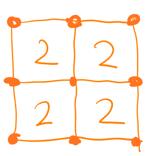


WARM-UP PROBLEM: Triangulate a 3x3 lattice

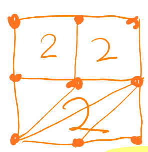
- How many triangles? 8
- How many different triangulations are possible?



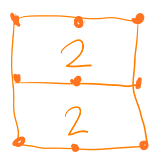
Count by which horizontal/vertical segments are present:



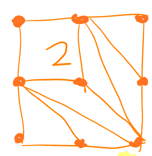
16 triangulations



8 × 4 = 32 triangulations



4 × 2 = 8 triangulations



2 × 4 = 8 triangulations

64 total triangulations

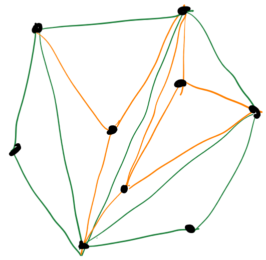
Extension:

18 triangles; 46,456 triangulations

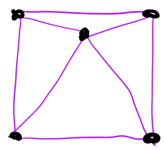
5x5 : 32 triangles; 736,983,568 triangulations

TRIANGULATION ALGORITHMS

- Triangle-Splitting Algorithm:



This triangulation cannot occur from triangle-splitting alg:



n = number of points

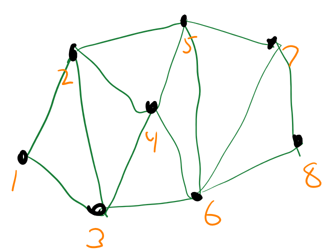
Complexity: Compute convex hull: $O(n \cdot \log n)$

triangulate hull: $O(n)$

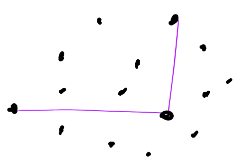
locate interior points: $O(n^2)$ or $O(n \log n)$ possible

$O(n^2)$ or $O(n \log n)$, based on

Incremental Algorithm



Does not produce all triangulation:



Complexity: Sort: $O(n \cdot \log n)$

Add n points, for each $\underbrace{\text{find visible part}}_{O(n)}$: $O(n^2)$

$n \cdot$

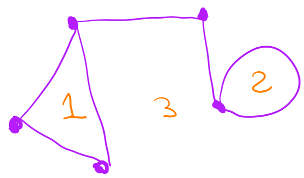
overall $O(n^2)$

EULER'S FORMULA

Let G be a connected planar graph with V vertices, E edges, and F faces (including the unbounded outer face).
Then:

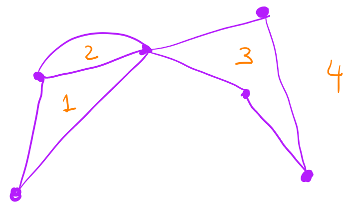
$$V - E + F = 2.$$

examples:



$V=5,\ E=6,\ F=3$

$V-E+F = 5-6+3 = 2$



$V=6,\ E=8,\ F=4$

$6-8+4 = 2$

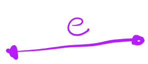
$V-E+F=2$

proof: use induction on the number of edges

base case: If $E=0$, then the graph is a isolated vertex. Then $V=1$ and $F=1$, and so $V-E+F = 1-0+1 = 2$.

induction: Assume the theorem holds for any graph with $E-1$ edges. Let graph G have E edges.

Choose any edge e of G.



→ If e connects two different vertices, then contract e, joins the two endpoints of e into one vertex.
This reduces the numbers of vertices and edges by 1 each.
This results in a graph with $V-1$ vertices, $E-1$ edges, F faces.

By induction hypothesis: $(V-\cancel{1})-(E-\cancel{1})+F = 2$

Thus: $V-E+F = 2.$

→ If e is a loop, then delete e, reducing the numbers of edges and faces by 1 each.

By induction hypothesis: $V-(E-\cancel{1})+(F-\cancel{1}) = 2$

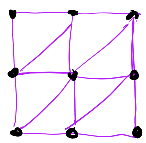
Thus: $V-E+F = 2.$

In either case, we have $V - E + F = 2$.

How many triangles in a triangulation of a point set?

THEOREM: Let S be a planar point set with not all points collinear. If S has h points on its convex hull and k points in its interior, then each triangulation has $2k+h-2$ triangles.

example:



$h = 8$
$k = 1$

$2k+h-2 = 2(1)+8-2 = 8$ triangles

$3k+2h-3 = 3(1)+2(8)-3 = 16$ edges

proof: Let $n = h+k$ and let t be the number of triangles. Then there are $t+1$ faces in the planar graph determined by the triangulation.

Count the edges:

- t triangles have $3t$ total edges
- unbounded exterior face has h edges.

Since each edge borders 2 faces, there are $\boxed{\frac{3t+h}{2}}$ edges.

Euler's formula says: $V - E + F = (h+k) - \frac{3t+h}{2} + (t+1) = 2$

$$2(h+k) - (3t+h) + 2(t+1) = 4$$
$$2h + 2k - 3t - h + 2t + 2 = 4$$
$$h + 2k - t = 2$$
$$h + 2k - 2 = t$$

Thus, there are $t = 2k+h-2$ triangles.

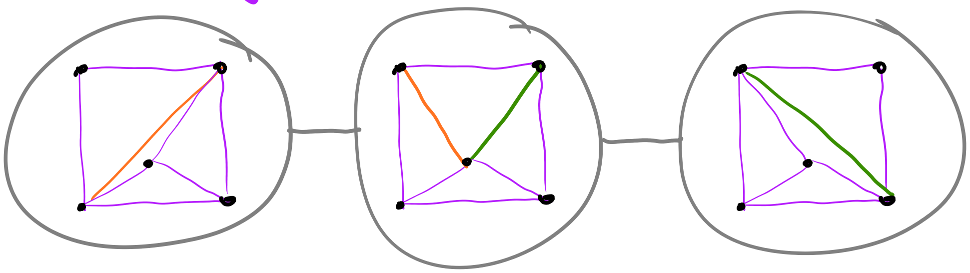
Also the number of edges is:

$$\frac{3t+h}{2} = \frac{3(h+2k-2)+h}{2} = \frac{3h+6k-6+h}{2} = \frac{4h+6k-6}{2}$$

$$E = 3k+2h-3$$

Sometimes, two triangulations differ by only one "edge flip".

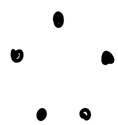
example:



FLIP GRAPH: For a point set S, the flip graph is a graph whose nodes are the set of triangulations of S. Two nodes T_1 and T_2 are connected by an edge if one diagonal of T_1 can be flipped to obtain T_2.

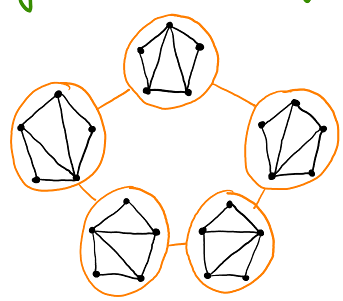
PROBLEMS: Find the flip graph of the following:

(a)

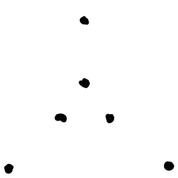


convex pentagon (5 triangulations)

flip graph:

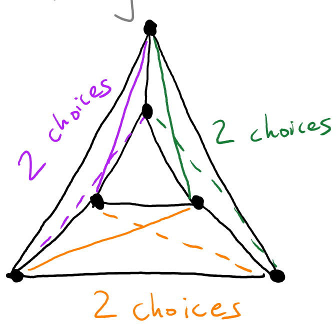


(b)

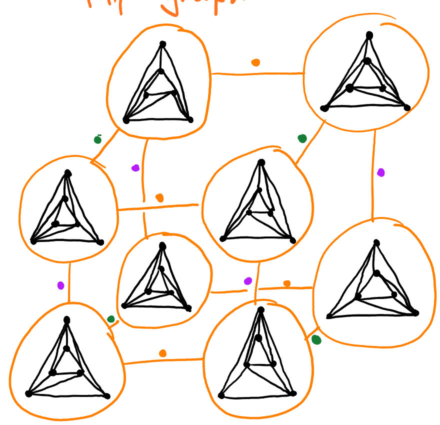


(8 triangulations)

8 triangulations:



flip graph:



cube!

QUESTION: Is the flip graph always connected?

Can every triangulation be obtained from every other by edge flips?

next time