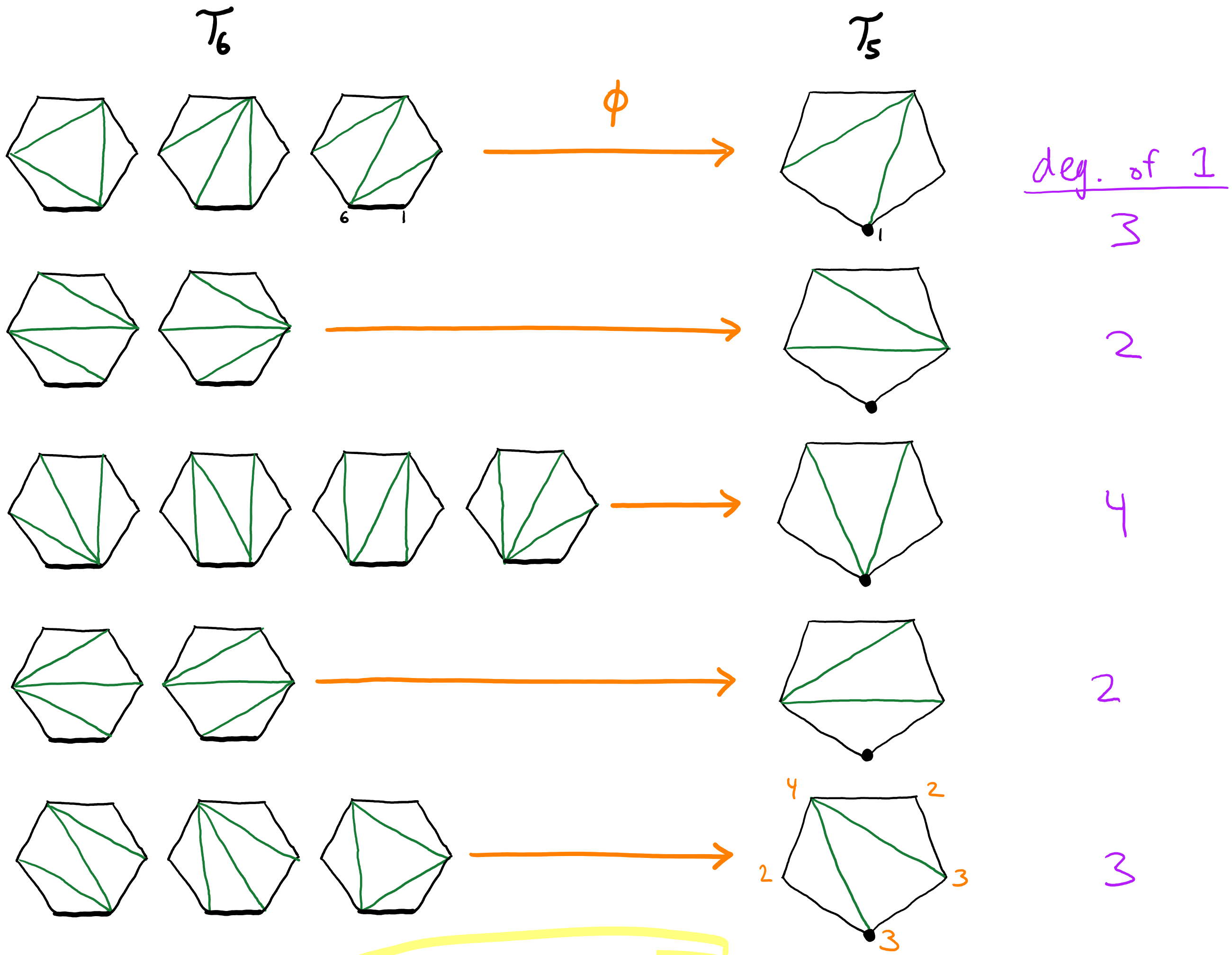
edge contraction example



$$t_6 = \sum_{T \in \mathcal{T}_5} \text{deg. of } 1 \text{ in } T$$

Now sum over all vertices of T of P_5

$$5 \cdot t_6 = \sum_{i=1}^{5} \left(\sum_{T \in \mathcal{T}_5} \text{deg. of vertex } i \text{ in } T \right)$$

$$= \sum_{T \in \mathcal{T}_5} \boxed{\sum_{i=1}^{5} \text{deg. of vertex } i \text{ in } T}$$

Sum of all degrees of vertices is twice the number of edges and diagonals of T

$2(5 \text{ edges} + 2 \text{ diagonals}) = (7)\,2$

$$5 \cdot t_6 = \sum_{T \in \mathcal{T}_5} 14 = 14 \cdot t_5$$

GENERALIZE: Now let $\phi: \mathcal{T}_{n+2}$ to $\mathcal{T}_{n+1}$

proof of number of triangulations of a convex n-gon

$$t_{n+2} = \sum_{T\in \mathcal{T}_{n+1}} \text{deg. of vertex } 1 \text{ in } T$$

$$(n+1)\cdot t_{n+2} = \sum_{j=1}^{n+1} \sum_{T\in \mathcal{T}_{n+1}} \text{deg. of vertex } j \text{ in } T$$

$$(n+1)\cdot t_{n+2} = \sum_{T\in \mathcal{T}_{n+1}} \boxed{\sum_{i=1}^{n+1} \text{deg. of } i \text{ in } T}$$

num. of triangulations is t_{n+1}

total deg. of triangulation T is twice the num. of edges and diagonals:
$2\big((n+1)+(n-2)\big) = 2(2n-1)$

$$\boxed{(n+1)\, t_{n+2} = 2(2n-1)\, t_{n+1}}$$

also $t_3 = 1$

$$t_{n+2} = \frac{2(2n-1)}{n+1}\, t_{n+1}$$

also: $t_{n+1} = \frac{2(2n-3)}{n}\, t_n$

Then:

$$t_{n+2} = \frac{2(2n-1)}{(n+1)}\cdot\frac{2(2n-3)}{(n)}\cdot\frac{2(2n-5)}{(n-1)}\cdot\frac{2(2n-7)}{(n-2)}\cdot \cdots \cdot \frac{2\cdot 3}{(3)}\,\cancel{t_3}\,1\cdot\frac{2}{2}$$

$$= \frac{1}{(n+1)!}\cdot\frac{2n\cdot(2n-1)\cdot(2(n-1))\cdot(2n-3)\cdot(2(n-2))\cdots(2\cdot 2)\,3\cdot 2}{n\cdot\ (n-1)\ \ (n-2)\ \cdots\ 2}$$

$$t_{n+2} = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\cdot\frac{(2n)!}{n!\,n!} = \frac{1}{n+1}\binom{2n}{n} = C_n$$

← Catalan number C_n

binomial coefficient "2n choose n"

Tetrahedron